Focus wave mode solutions of the inhomogeneous n-dimensional scalar wave equation

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In the following focus wave mode solutions of the inhomogeneous n-dimensional scalar wave equation are determined when the source term is of separable type (i.e., separable in the characteristic variables). In this case the n-dimensional inhomogeneous wave equation is transformed into a formally equivalent $(n-1)$-dimensional inhomogeneous diffusion equation having a complex longitudinal space-time independent variable. The unbounded space propagator of this diffusion equation and the Palmer-Donnelly line source term generalized to n dimensions are used to obtain inhomogeneous wave equation solutions in two, three, and four space dimensions. Localization of these solutions is shown to increase as the dimensionality increases. An infinitely long line source with finite radius is also considered. As the source radius increases past a certain point, the localization and amplitude of the central peak decrease dramatically.

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I. INTRODUCTION

Until recently focus wave mode (FWM) or localized wave (LW) solutions of the scalar wave [1–9], damped wave [10–12], Klein-Gordon [9–14], London [15,16], and Maxwell's [17–19] equations have been considered, primarily when solving homogeneous partial differential equations (PDEs). However, several attempts to solve the scalar inhomogeneous wave equation to provide physical insight into launching mechanisms for such waves have been made also [20–22]. These attempts have been restricted to three (space) dimensions only.

In this paper we obtain explicit FWM solutions (when n is two, three, and four) to the inhomogeneous n-dimensional scalar wave equation having a source term of a given form. When the source term is of separable type (separable referring to the characteristic variables), the n-dimensional inhomogeneous wave equation can be transformed to a PDE which is formally equivalent to a $(n-1)$-dimensional inhomogeneous diffusion (or heat) equation with a complex longitudinal space-time independent variable. The diffusion equation has a decided advantage over the wave equation in the following way. Its propagator in unbounded space exhibits no important formal differences between one, two, and n dimensions. Furthermore, the diffusion equation propagator is simply the envelope of the fundamental Gaussian LW solution of the scalar wave equation and thus the fundamental Gaussian solution can now be generalized to n dimensions. This propagator is used to determine an appropriate unbounded space Green's function which can then be integrated (along with the source term) to obtain solutions to the inhomogeneous scalar wave equation. The Palmer-Donnelly source term [20] for the inhomogeneous wave equation is used in two, three, and four dimensions to compare the effects of dimensionality on the LW solutions. This method offers an advantage over direct solution of the inhomogeneous wave equation: only a retarded Green's function need be used, whereas the direct method utilizes both retarded and advanced Green's function components [20,21]. Thus in the present method problems with causality are, hopefully, not as likely to occur.

In Sec. II, we transform the n-dimensional wave equation into a $(n-1)$-dimensional diffusion equation equivalent. In Sec. III, we use the unbounded space propagator and the Palmer-Donnelly source term generalized to n dimensions to obtain inhomogeneous scalar wave equation solutions in two, three, and four dimensions. In Sec. IV we use an infinitely long line source with finite radius (which is the idealization of a finite radius long wire with current very close to the surface) to achieve a LW solution to the inhomogeneous wave equation. Section V contains the conclusions.

II. THE n-DIMENSIONAL SCALAR WAVE EQUATION

We consider the n-dimensional scalar homogeneous wave equation in Cartesian coordinates to be given by

$$\left[\nabla_n^2-\frac{\partial^2}{\partial x_0^2}\right]\Psi(\vec{r}_n,x_0)=0\ , \tag{1}$$

where

$$\nabla_n^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\ , \tag{2}$$

$x_0=ct$, n can be any integer, and

$$\vec{r}_n=(x_1,x_2,\ldots,x_n). \tag{3}$$

Using the characteristic variable substitution

$$\zeta=x_n-x_0\ ,\quad \eta=x_n+x_0\ , \tag{4}$$

Eq. (1) becomes

$$\left[\nabla_{n-1}^2+4\frac{\partial^2}{\partial\zeta\partial\eta}\right]\Psi(\vec{r}_{n-1},\zeta,\eta)=0\ . \tag{5}$$

If the solution, $\Psi(\vec{r}_{n-1},\zeta,\eta)$, is assumed to have the form

$$\Psi(\vec{r}_{n-1},\zeta,\eta)=f(\vec{r}_{n-1},\zeta)e^{i\beta\eta} , \tag{6}$$

then (5) becomes

$$\left[\nabla^2_{n-1}+4i\beta\frac{\partial}{\partial\zeta}\right]f(\vec{r}_{n-1},\zeta)=0 , \tag{7}$$

and once $f(\vec{r}_{n-1},\zeta)$ is found, Ψ is immediately known. In its present form, Eq. (7) is formally equivalent to a Schrödinger equation, and further transformation, i.e.,

$$\tau=a_1+i\zeta , \tag{8}$$

reduces (7) to the equivalent form of a diffusion (or heat) equation,

$$\left[\nabla^2_{n-1}-4\beta\frac{\partial}{\partial\tau}\right]f(\vec{r}_{n-1},\tau)=0 . \tag{9}$$

For the specific transformation (8), a_1 must be real and positive while $\zeta=x_n-ct$ is real also. Thus τ is a complex space-time independent variable with one "longitudinal" spatial dimension x_n separated from the remaining "transverse" spatial variables. In (9), $1/4\beta$ acts like a constant diffusion coefficient.

In the inhomogeneous wave equation case,

$$\left[\nabla^2_n-\frac{\partial^2}{\partial x_0^2}\right]\Psi(\vec{r}_n,x_0)=s(\vec{r}_n,x_0) , \tag{10}$$

if the source term has the general form, i.e., separable in the characteristic variables ζ and η

$$s(\vec{r}_n,x_0)=g(\vec{r}_{n-1},\zeta)e^{i\beta\eta} , \tag{11}$$

then, using (8), (10) can be rewritten as

$$\left[\nabla^2_{n-1}-4\beta\frac{\partial}{\partial\tau}\right]f(\vec{r}_{n-1},\tau)=g(\vec{r}_{n-1},\tau) , \tag{12}$$

an $(n-1)$-dimensional inhomogeneous diffusion equation, which does not explicitly involve η.

Returning to the homogeneous n-dimensional diffusion equation in (9), we can use the unbounded space propagator of this equation to obtain solutions of both the homogeneous and inhomogeneous n-dimensional wave equation. The diffusion propagator $K_0^{(n-1)}(\vec{r}_{n-1},\tau;\ \vec{r}'_{n-1},\tau')$ in unbounded space is a solution of the homogeneous PDE, given in (9), that satisfies the following conditions [23]:

$$\left[\nabla^2_{n-1}-4\beta\frac{\partial}{\partial\tau}\right]K_0^{(n-1)}(\vec{r}_{n-1},\tau;\ \vec{r}'_{n-1},\tau')=0 \quad \text{for } |\tau|>|\tau'| , \tag{13a}$$

$$K_0^{(n-1)}(\vec{r}_{n-1},\tau';\ \vec{r}'_{n-1},\tau')=\delta(\vec{r}_{n-1}-\vec{r}'_{n-1}) , \tag{13b}$$

$$K_0^{(n-1)}\rightarrow 0 \text{ as } |\vec{r}_{n-1}|\rightarrow\infty . \tag{13c}$$

It is well known that the propagator for (9) is given by [23]

$$K_0^{(n-1)}(\vec{R},\xi)=\left[\frac{\beta}{\pi\xi}\right]^{(n-1)/2}\exp\left[\frac{-\beta\vec{R}^2}{\xi}\right] , \tag{14}$$

where

$$\vec{R}=\vec{r}_{n-1}-\vec{r}'_{n-1} \tag{15a}$$

and

$$\xi=\tau-\tau'=(a_1-a'_1)+i(\zeta-\zeta') . \tag{15b}$$

As long as $|\tau|>|\tau'|$ so that $|\xi|>0$, (14) is a solution to the $(n-1)$-dimensional homogeneous diffusion equation in unbounded space, and thus

$$L_0^{(n)}(\vec{R},\xi)=K_0^{(n-1)}(\vec{R},\xi)e^{i\beta(\eta-\eta')} \tag{16}$$

is a solution to the n-dimensional homogeneous wave equation in unbounded space. Specifically the three-dimensional ($n=3$) wave equation has the solution

$$L_0^{(3)}(\vec{R},\xi)=K_0^{(2)}(\vec{R},\xi)e^{i\beta(\eta-\eta')} , \tag{17}$$

or, assuming that $\vec{r}'_2=\tau'=\eta'=0$,

$$L_0^{(3)}(\rho,\tau)=\left[\frac{\beta}{\pi(a_1+i\zeta)}\right]\exp\left[\frac{-\beta\rho^2}{(a_1+i\zeta)}+i\beta\eta\right] \tag{18}$$

(since $\vec{r}_2^{\,2}=x_1^2+x_2^2=\rho^2$, $\tau=a_1+i\zeta$). Equation (18) is a focus-wave-mode solution of the three-dimensional homogeneous wave equation, and it is well known as the fundamental Gaussian [3]. The fundamental Gaussian is indeed fundamental in the sense that [excluding the $\exp(i\beta\eta)$ factor], the diffusion equation propagator *is* the fundamental Gaussian solution. Furthermore, once the propagator (of one lower dimension) is known, any initial value problem involving the diffusion equation may be solved, and this solution can become a solution of the wave equation of one higher dimension. For example, if we wish to solve (9) (with $n=3$) subject to the initial condition that

$$f(\vec{r}_2,\tau_0)=a(|\vec{r}_2|) , \tag{19}$$

where $|\vec{r}_2|=\rho$, $\tau_0=a_0+i(z_0-ct_0)$, then for $|\tau|>|\tau_0|$,

$$f(\rho,\tau)=\int_{-\infty}^{\infty}d\rho'\rho'K_0^{(2)}(\rho,\tau;\rho',\tau_0)a(\rho') , \tag{20}$$

which gives the solution to (9) subject to (19).

Also in the absence of boundaries, we would like to solve the n-dimensional inhomogeneous wave equation, provided its source term is separable in the characteristic variables and thus obeys (11). When the source term obeys (11), we can use the fact that the propagator of the $(n-1)$-dimensional diffusion equation is closely related to the Green's function by

$$G_0^{(n-1)}(\vec{r}_{n-1},\tau;\vec{r}'_{n-1},\tau')=h(\tau-\tau')K_0^{(n-1)}(\vec{r}_{n-1},\tau;\ \vec{r}'_{n-1},\tau') , \tag{21}$$

where $h(\tau-\tau')$ is the usual step function.

If $g(\vec{r}_{n-1},\tau)$ is the source term of the diffusion equation as in (12), the general solution to the $(n-1)$-dimensional inhomogeneous diffusion equation is

$$f(\vec{r}_{n-1},\tau)=\int_{-\infty}^{\tau^+}d\tau'\int dV' G_0^{(n-1)}(\vec{r}_{n-1},\tau;\vec{r}\,'_{n-1},\tau')\times g(\vec{r}\,'_{n-1},\tau')\ , \tag{22}$$

in the absence of boundaries and for zero initial conditions with dV' as the source volume and $\tau^+=\tau+\varepsilon$, $\lim_{\varepsilon\to 0}\tau^+=\tau$. Usually this limit is unnecessary and if no sources exist and act before some initial space-time point τ_0 then (22) can be written as [23]

$$f(\vec{r}_{n-1},\tau)=\int_{\tau_0}^{\tau}d\tau'\int dV' K_0^{(n-1)}(\vec{r}_{n-1},\tau;\vec{r}\,'_{n-1},\tau')\times g(\vec{r}\,'_{n-1},\tau')\ ,\quad |\tau|\geq|\tau'|\ , \tag{23}$$

and the step function is unity in this region. Of course, for $|\tau|<|\tau'|$, $f(\vec{r}_{n-1},\tau)$ would be zero, thus carrying the step function along is unnecessary.

In Sec. III we will use (23) to compute some focus-wave-mode solutions of the inhomogeneous wave equation for two, three, and four dimensions.

III. FOCUS WAVE MODES AND DIMENSIONALITY

We assume a generalization of the infinitely long, infinitesimally thin line source in free space introduced by Palmer and Donnelly [20]. In n dimensions, we set

$$s(\vec{r}_n,x_0)=\frac{\delta(\vec{r}_{n-1})}{(a_1+i\zeta)}\exp(i\beta\eta)\ , \tag{24}$$

which satisfies (11). From Sec. II, we can write the solution to (12) as

$$f(\vec{r}_{n-1},\tau)=\int_{\tau_0}^{\tau}d\tau'\int dV' K_0^{(n-1)}(\vec{r}_{n-1},\tau;\vec{r}\,'_{n-1},\tau')\times\frac{\delta(\vec{r}\,'_{n-1})}{\tau'}\ . \tag{25}$$

Substituting (14) into (25), we have

$$f(\vec{r}_{n-1},\tau)=\left[\frac{\beta}{\pi}\right]^{(n-1)/2}\times\int_{\tau_0}^{\tau}d\tau'\int dV'\left[\frac{1}{\xi}\right]^{(n-1)/2}\exp\left[-\frac{\beta\vec{R}^2}{\xi}\right]\times\frac{\delta(\vec{r}\,'_{n-1})}{\tau'}\ , \tag{26}$$

with ξ and $\vec{R}$ as previously.

The most important case is $n=3$. When $n=3$, (26) becomes

$$f(\rho,\tau)=\left[\frac{\beta}{\pi}\right]\int_{\tau_0}^{\tau}d\tau'\int\rho' d\rho' d\phi'\exp\left[\frac{-\beta|\vec{\rho}-\vec{\rho}\,'|^2}{(\tau-\tau')}\right]\times\frac{\delta(\rho')}{(2\pi)\rho'\tau'(\tau-\tau')}\ , \tag{27}$$

or after the δ-function integration,

$$f(\rho,\tau)=\left[\frac{\beta}{\pi}\right]\int_{\tau_0}^{\tau}d\tau'\frac{\exp\left[\frac{-\beta\rho^2}{(\tau-\tau')}\right]}{\tau'(\tau-\tau')}\ . \tag{28}$$

Using several variable transformations, (28) can be written as

$$f(\rho,\tau)=\frac{-\beta}{\pi\tau}\int_{u_0}^{\infty}du\frac{\exp\left[\frac{-\beta\rho^2 u}{\tau}\right]}{(1-u)}\ , \tag{29}$$

where $u_0=\tau/(\tau-\tau_0)$.

Upon integration [24], (29) becomes

$$f(\rho,\tau)=\frac{-\beta}{\pi\tau}\exp\left[\frac{-\beta\rho^2}{\tau}\right]\mathrm{Ei}\left[\frac{\beta\rho^2}{\tau}\left[1-\frac{\tau}{\tau-\tau_0}\right]\right]\ , \tag{30}$$

where Ei is the exponential integral.

Using $-\mathrm{Ei}(x)=E_1(-x)-i\pi$, letting $\tau_0\to-\infty$, and using (6),

$$\Psi(\rho,\tau)=\left[\frac{\beta}{\pi}\right]\frac{\exp\left[\frac{-\beta\rho^2}{\tau}\right]}{\tau}\left[E_1\left[\frac{-\beta\rho^2}{\tau}\right]-i\pi\right]\times\exp(i\beta\eta)\ , \tag{31}$$

which is essentially the Palmer-Donnelly solution [20] of the wave equation, except for the constants and the fact that a step function is missing. It is somewhat surprising in the Palmer-Donnelly solution that the step function occurs as an additive term rather than a product term [21]. However, for our method, we have an explanation as to why the step function occurs in the Palmer-Donnelly solution and not in (31). They state that their Green's function [20] [Ref. [20], Eq. (14)] contains both retarded and advanced components and the step function must be used to ensure that their formulation is causal. However, using the diffusion equation allows us to have a Green's function with a retarded component only and thus, as shown by Eqs. (22) and (23), the step function is not necessary in this capacity. Equation (31) can also be obtained (with arbitrary constants) as a localized wave solution of the second kind to the homogeneous three-dimensional scalar wave equation [11] [Ref. 11, Eq. (30)] when $q=-1$. Obtaining it via the line source in (24) gives the solution much more physical significance [21] (see Fig. 1).

When the dimensionality of the wave equation is two, the line source in two dimensions is

$$s(\vec{r}_2,x_0)=\frac{\delta(x)}{\tau}\exp(i\beta\eta)\ , \tag{32}$$

and thus

$$f(x,\tau)=\left[\frac{\beta}{\pi}\right]^{1/2}\int_{\tau_0}^{\tau}d\tau'\int dx'\frac{\exp\left[\frac{-\beta(x-x')^2}{(\tau-\tau')}\right]}{\tau'\sqrt{\tau-\tau'}}\times\delta(x')\ . \tag{33}$$

After the δ-function integration and several variable transformations

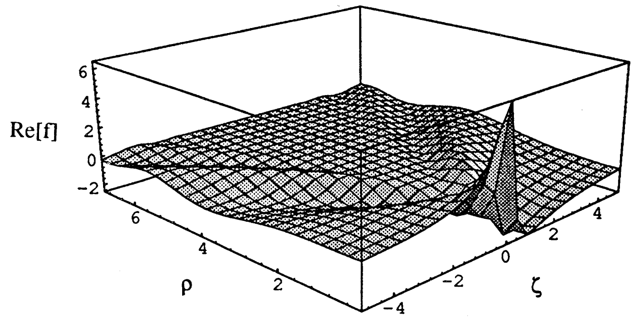

FIG. 1. Focus-wave-mode solution envelope of the three-dimensional wave equation assuming an infinite line source of separable type ($\beta=1$, $a_1=0.5$).

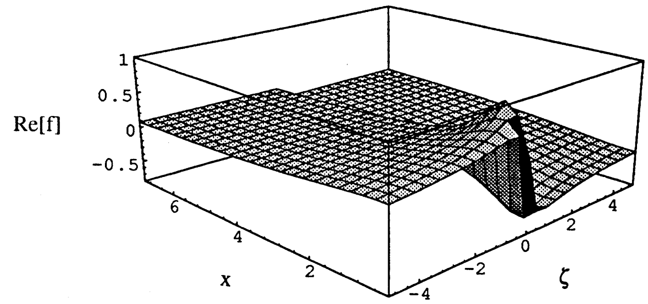

FIG. 2. Focus-wave-mode solution envelope of the two-dimensional wave equation assuming an infinite line source of separable type ($\beta=1$, $a_1=0.5$).

$$f(x,\tau)=-i\left[\frac{\beta}{\pi\tau}\right]^{1/2}\int_0^\infty \frac{du\,\exp\left[\frac{\beta x^2 u}{\tau}\right]}{\sqrt{u}\,(u+1)}\;, \tag{34}$$

where we have let $\tau_0\rightarrow-\infty$, and (34) becomes [25]

$$f(x,\tau)=-i\left[\frac{\pi\beta}{\tau}\right]^{1/2}\exp\left[\frac{-\beta x^2}{\tau}\right]\times\text{erfc}\left[\left[-\frac{\beta x^2}{\tau}\right]^{1/2}\right]\;, \tag{35}$$

where erfc is the complementary error function. Equation (35) (when the $e^{i\beta\eta}$ factor is included) gives a localized wave two-dimensional solution to the inhomogeneous wave equation when the source term is a line source. The real part of $f(x,\tau)$ is plotted in Fig. 2.

The case of the four-dimensional wave equation is very interesting also. Let

$$s(\vec{r}_4,x_0)=\frac{\delta(r)}{4\pi r^2\tau}\exp(i\beta\eta)\;, \tag{36}$$

where $|\vec{r}_3|=(x_1^2+x_2^2+x_3^2)^{1/2}=r$. Using standard spherical coordinates for this case,

$$f(\vec{r},\tau)=\left[\frac{\beta}{\pi}\right]^{3/2}\int_{\tau_0}^{\tau}d\tau'\int_0^{2\pi}\int_0^{\pi}\int_0^{r}r'^2\sin\theta' dr'd\theta'd\phi'\left[\frac{1}{(\tau-\tau')}\right]^{3/2}\exp\left[\frac{-\beta|\vec{r}-\vec{r}\,'|^2}{(\tau-\tau')}\right]\frac{\delta(r')}{4\pi r'^2\tau'}\;, \tag{37}$$

or (37) becomes

$$f(\vec{r},\tau)=\left[\frac{\beta}{\pi}\right]^{3/2}\int_{\tau_0}^{\tau}\frac{\exp\left[\frac{-\beta r^2}{\tau-\tau'}\right]d\tau'}{\tau'(\tau-\tau')^{3/2}}\;. \tag{38}$$

As previously with $\tau_0\rightarrow-\infty$, (38) can be put into the form

$$f(\vec{r},\tau)=\left[\frac{\beta}{\pi\tau}\right]^{3/2}\int_0^\infty dy\frac{y^2}{(y^2-1)}\exp\left[\frac{-\beta r^2y^2}{\tau}\right]\;, \tag{39}$$

which integrates to [24]

$$f(\vec{r},\tau)=\frac{\beta}{\pi\tau r}-\frac{i}{\pi}\left[\frac{\beta}{\tau}\right]^{3/2}\exp\left[\frac{-\beta r^2}{\tau}\right]\times\text{erfc}\left[\left[\frac{-\beta r^2}{\tau}\right]^{1/2}\right]\;. \tag{40}$$

When $f(\vec{r},\tau)$ in (40) is multiplied by $e^{i\beta\eta}$, we have a localized wave or focus-wave-mode solution to the four-dimensional inhomogeneous wave equation. The real part of (40) is shown in Fig. 3.

A comparison of Figs. 1, 2, and 3 indicates that the localization property of the real part of the envelopes, $f(\vec{r}_{n-1},\tau)$, is greatly increased as the dimensionality n increases. Using the same parameters in each plot $a_1=0.5$ and $\beta=1$, the two-dimensional solution extends along the transverse direction and is not very localized (focused near $x=0$) at all. The three-dimensional solution is much better, peaking at $\zeta=0$, ρ very close to zero, with some much smaller magnitude ripples as ρ increases. The four-dimensional solution is extremely localized near $r=0$ and is the best LW solution of the three.

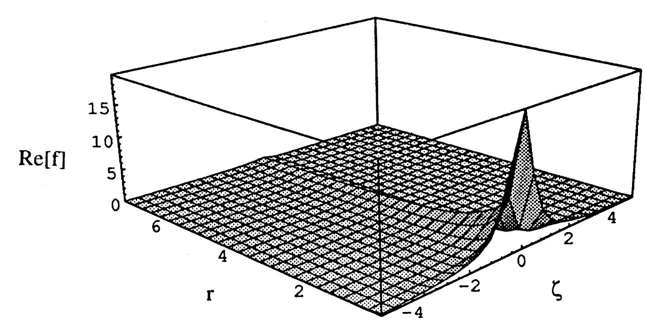

FIG. 3. Focus-wave-mode solution envelope of the four-dimensional wave equation assuming an infinite line source of separable type ($\beta=1$, $a_1=0.5$).

IV. LINE SOURCE WITH FINITE RADIUS

When the dimensionality of the wave equation is three, and we assume an infinitely long source with a finite radius a

$$s(\vec{r}_3,x_0)=\frac{\delta(\rho-a)}{2\pi\rho\tau}\exp(i\beta\eta)\ , \tag{41}$$

where $|\vec{r}_2|=(x_1^2+x_2^2)^{1/2}=\rho$, the solution to the inhomogeneous diffusion equation becomes

$$f(\rho,\tau)=\frac{\beta}{\pi}\int_{\tau_0}^{\tau}d\tau'\int_0^a\int_0^{2\pi}\rho' d\rho' d\phi' \times\exp\left[\frac{-\beta|\vec{\rho}-\vec{\rho}'|^2}{(\tau-\tau')}\right]\times\frac{\delta(\rho'-a)}{2\pi\tau'(\tau-\tau')\rho'}\ . \tag{42}$$

Performing the δ-function integration first,

$$f(\rho,\tau)=\frac{\beta}{2\pi^2}\int_{\tau_0}^{\tau}d\tau'\int_0^{2\pi}d\phi'\frac{\exp\left[\dfrac{-\beta(\rho^2+a^2-2\rho a\cos\phi')}{(\tau-\tau')}\right]}{\tau'(\tau-\tau')}\ . \tag{43}$$

Considering the ϕ' integral, we get

$$I_\phi=\int_0^{2\pi}d\phi'\exp\left[\frac{2\beta\rho a\cos\phi'}{(\tau-\tau')}\right] =2\pi I_0\left[\frac{2\beta\rho a}{\tau-\tau'}\right]\ , \tag{44}$$

where I_0 is a modified Bessel function. Using (44) in (43),

$$f(\rho,\tau)=\frac{\beta}{\pi}\int_{\tau_0}^{\tau}d\tau'\frac{\exp\left[\dfrac{-\beta(\rho^2+a^2)}{\tau-\tau'}\right]}{\tau'(\tau-\tau')}I_0\left[\frac{2\beta\rho a}{\tau-\tau'}\right]\ . \tag{45}$$

When $\beta\rho a/\tau$ is small, we can use the series form of I_0, i.e.,

$$I_0(x)=\sum_{k=0}^{\infty}\frac{(x^2/4)^k}{(k!)^2}\ , \tag{46}$$

or (45) becomes (allowing $\tau_0\rightarrow-\infty$)

$$f(\rho,\tau)=\frac{\beta}{\pi\tau}\sum_{n=0}^{\infty}\frac{\left[\left[\dfrac{\beta\rho a}{\tau}\right]^2\right]^n}{(n!)^2}\times\int_0^{\infty}dx\frac{\exp\left[\dfrac{-\beta(\rho^2+a^2)x}{\tau}\right]}{(x-1)}x^{2n}\ . \tag{47}$$

Using $-\text{Ei}(x)=E_1(-x)-i\pi$, upon integration, (47) becomes (Ref. [24], No. 3.353.5)

$$f(\rho,\tau)=\frac{\beta}{\pi\tau}\sum_{n=0}^{\infty}\frac{\left[\dfrac{\beta\rho a}{\tau}\right]^{2n}}{(n!)^2}\left\{\exp(-w)[E_1(-w)-i\pi]-\sum_{k=1}^{2n}(k-1)!(w)^{-k}\right\}\ , \tag{48}$$

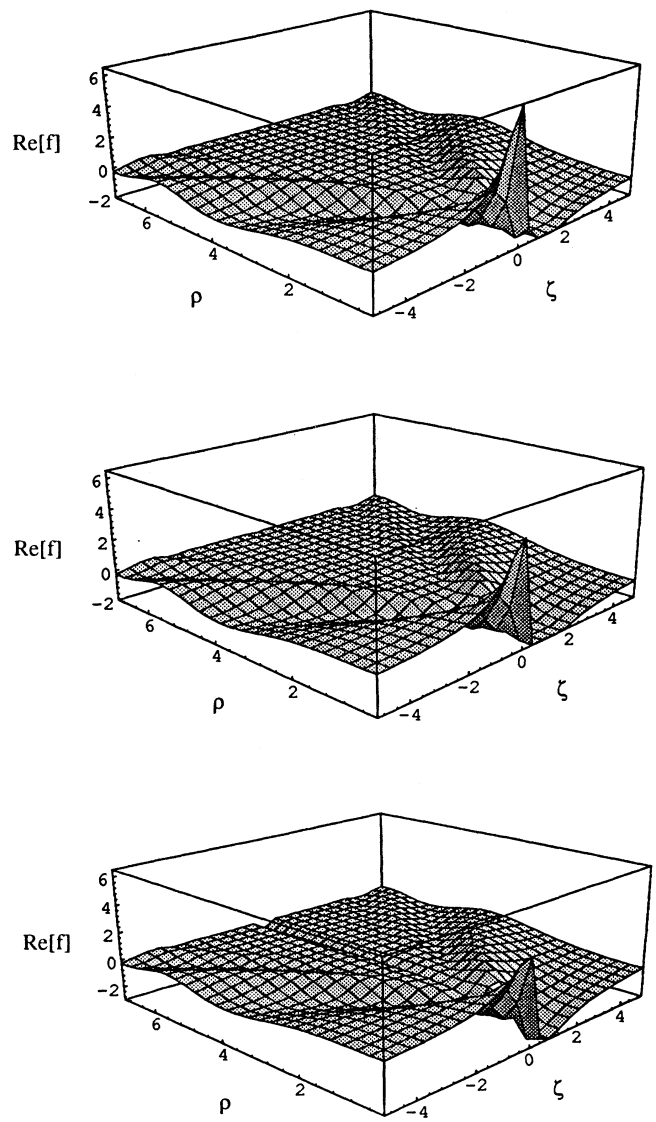

FIG. 4. Focus-wave-mode solution envelope of the three-dimensional wave equation assuming a finite radius infinitely long source of separable type ($\beta=1$, $a_1=0.5$). (a) $\beta a=0.01$, (b) $\beta a=0.075$, (c) $\beta a=0.15$.

where

$$w = \frac{\beta(\rho^2 + a^2)}{\tau} \ . \quad (49)$$

The real part of (48) is plotted versus ρ and ζ in Fig. 4 for various values of the source radius, a. When a is still small, i.e., $a = 0.01$, the finite radius source and the three-dimensional infinitesimal line source shown in Fig. 1 are almost identical. As a increases to 0.075, the magnitude of the pulse is smaller and a "trough" is beginning to form just behind the pulse peak. Finally for $a = 0.15$, the pulse peak is much smaller and less localized. Thus the localization is not very good as a continues to increase.

V. CONCLUSIONS

We have obtained localized wave solutions to the n-dimensional inhomogeneous scalar wave equation provided the source term has a specific form which is separable in the characteristic variables, allowing reduction to an equivalent inhomogeneous $(n-1)$-dimensional diffusion equation. Two-, three-, and four-dimensional inhomogeneous solutions were determined, and localization is shown to increase as the dimensionality increases. A finite radius line source in three dimensions was considered also. Its localization amplitude characteristics are dependent on the magnitude of the source radius.

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